L^1-stability of periodic stationary solutions of scalar convection-diffusion equations

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Abstract

The aim of this paper is to study the L^1-stability of periodic stationary solutions of scalar convection-diffusion equations. We obtain dispersion in L^2 for all space dimensions using Kružkov type entropy. And when the space dimension is one, we estimate the number of sign changes of a solution to obtain L^1-stability.

Keyword : L^1-stability, periodic stationary solutions, entropy, dispersion inequality, lap number.

1 Introduction

We study the solutions of a scalar convection-diffusion equation of the form:

$$\partial_t u + \operatorname{div}(f(u,x)) = \Delta u, \quad t > 0, x \in \mathbb{R}^d, \tag{1}$$

where $x \mapsto f(\cdot, x)$ is an Y-periodic function with $Y = \prod_{i=1}^{d}(0, T_i)$ the basis of a lattice. We assume that f belongs to $\mathcal{C}^2(\mathbb{R}, \mathcal{C}^1(\mathbb{R}^d))$. For this equation, periodic stationary solutions w_p exist and are parameterized by their space average p: this is a result of Dalibard in [2]. In this paper, we focus on the L^1-stability of these periodic stationary solutions.

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When f only depends on u, the periodic stationary solutions are the constants and the L^1-stability of the constants is already proved by Freistühler and Serre in the one-dimensional space case in [3] and by Serre in all space dimension in [9]. We define the space

$$L_0^1(\mathbb{R}^d) = \{u \in L^1(\mathbb{R}^d) : \int_{\mathbb{R}^d} u(x)dx = 0\}.$$

With this notation, the result can be written as follows:

Theorem 1. [9] *For all $k \in \mathbb{R}, b \in L_0^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, the unique solution $u \in L^\infty_{loc}(\mathbb{R}, L^\infty(\mathbb{R}^d))$ of*

$$\begin{cases} \partial_t u + \operatorname{div}(f(u)) = \Delta u, & t > 0, x \in \mathbb{R}^d, \\ u(0,x) = k + b(x), & x \in \mathbb{R}^d, \end{cases} \tag{2}$$

satisfies:

$$\lim_{t\to\infty} \|u(t,\cdot) - k\|_1 = 0.$$

The proof of this result can be made in 3 steps. First, the global existence of solution of (2) is proved using the Duhamel's formula with $\operatorname{div}(f(u))$ as a perturbation of the Heat equation, one obtains :

$$u(t) = K^t * u_0 + \int_0^t \operatorname{div} K^{t-s} * f(u(s))ds.$$

The maximum principle allows to conclude about global existence by induction. This defines the nonlinear semigroup $\tilde{S}^t$ so that $u(t) = \tilde{S}^t u_0$ is the solution of (2).

Secondly, one establishes the so-called four "Co-properties" for u_0, v_0 in $L^\infty(\mathbb{R}^d)$:

1. Comparison: $u_0 \leq v_0$ a.e.$\Rightarrow \tilde{S}^t u_0 \leq \tilde{S}^t v_0$ a.e.,
2. Contraction: $v_0 - u_0 \in L^1(\mathbb{R}^d) \Rightarrow \tilde{S}^t v_0 - \tilde{S}^t u_0 \in L^1(\mathbb{R}^d)$ and
$$\|\tilde{S}^t v_0 - \tilde{S}^t u_0\| \leq \|v_0 - u_0\|,$$
3. Conservation (of mass): $v_0 - u_0 \in L^1(\mathbb{R}^d) \Rightarrow \tilde{S}^t v_0 - \tilde{S}^t u_0 \in L^1(\mathbb{R}^d)$ and
$$\int_{\mathbb{R}^d} (\tilde{S}^t v_0 - \tilde{S}^t u_0) = \int_{\mathbb{R}^d} (v_0 - u_0),$$

4. Constants: if u_0 is a constant, then $\tilde{S}^t u_0 \equiv u_0$.

Two methods allow to conclude: one in one space dimension and another one in all space dimension. The first one is due to Freistühler and Serre [3]: they study the number of sign changes of the solution. Having assumed that $k = 0, f(0) = 0$, they study the primitive V of the solution u which vanishes at $-\infty$: $V(x,t) = \int_{-\infty}^{x} u(y,t)dy$. Since $b \in L^1_0(\mathbb{R})$, this primitive also vanishes at $+\infty$ and belongs to $L^\infty(\mathbb{R})$. Moreover, V satisfies a parabolic equation

$$\partial_t V + f(\partial_x V) = \partial_x^2 V.$$

They also apply the lemma of Matano [4] on V to estimate the number of sign changes of the derivative of V: u. Estimates on both $\|u(t)\|_{L^1}$ by $\|V(t)\|_{L^\infty}$ follow. Using L^2-estimates on the equations on both u and V, one shows that $\lim_{t\to\infty} \|V(t)\|_{L^\infty} = 0$, which permits to obtain the theorem.

The second method, due to Serre [9], is based on the Duhamel's formula. A dispersion inequality is obtained using the entropy $u \mapsto u^2$ for equation (2) and L^1-contraction, one obtains :

$$\|\tilde{S}^t u_0\|_2 \leq c_d \frac{\|u_0\|_1}{t^{d/4}}.$$

Under the rather general assumption that $f(u)$ is bounded by $|u|^2$,we prove $\lim_{t\to\infty} \|\tilde{S}^t b\|_1 = 0$ combining dispersion estimate and estimates on the heat kernel.

In this article, we will see how we can adapt some of these arguments to the case where f depends both on u and x. We recall that in this case the stationary solutions w_p considered are periodic, parameterized by their space average p.

We obtain one theorem in the one-dimensional space case :

Theorem 2. *For all $p \in \mathbb{R}, b \in L^1_0(\mathbb{R}) \cap L^\infty(\mathbb{R})$, the unique solution u in $L^\infty_{loc}(\mathbb{R}, L^\infty(\mathbb{R}))$ of*

$$\begin{cases} \partial_t u + \operatorname{div}(f(u,x)) = \Delta u, & t > 0, x \in \mathbb{R}, \\ u(0,x) = w_p + b(x), & x \in \mathbb{R}, \end{cases}$$

satisfies:

$$\lim_{t\to\infty} \|u(t,\cdot) - w_p\|_1 = 0.$$

First, we observe that in this theorem we assume $\int_{-\infty}^{\infty} b(x)dx = 0$. This assumption is necessary because of the conservation of mass:

$$\int_{\mathbb{R}^d}(v - w_p) = \int_{\mathbb{R}^d}(v_0 - w_p) = \int_{\mathbb{R}^d} b.$$

Actually, we can not have L^1-convergence when $\int_{\mathbb{R}^d} b \neq 0$. But this assumption is not necessary to prove L^p-convergence for $1 < p \leq 2$ and in this case we obtain a rate of convergence $d/2(1 - 1/p)$.

To prove the theorem, we use results on the nonlinear semigroup and the lemma of Matano, as in [3]. The main difference with the proof of Serre and Freistühler ([9] & [3]) appears in the proof of L^2-estimates for u and its primitive V. Since the problem is inhomogeneous, $u \mapsto u^2$ is not an entropy and we have to find a new entropy to prove dispersion inequality. For V the results on periodic stationary solutions of Dalibard permit to prove that $\|V\|_2$ is bounded.

The paper is organized as follows. In section 2, we recall the result obtained by Dalibard in [2] about the existence of periodic stationary solutions. In section 3, we focus on the existence and the properties of our nonlinear semigroup in all space dimension: comparison principle, contraction in L^1, conservation of mass, dispersion inequality. For its existence and its three first properties the proofs are similar to the homogeneous case $f(u, x) = f(u)$, except that the maximum principle does not hold anymore and is replaced by a comparison principle. For the dispersion inequality, we build a new type of Kružkov entropy, based on periodic stationary solutions instead of constants. In section 4, we focus on the one-dimensional space case, and prove theorem 2 using the lemma of Matano about the number of sign changes.

2 Existence of stationary solutions

In this section, we recall the existence result of Dalibard [2]. When f depends only on u, but not on x, i.e. when we are in the case studied by Serre in [9], the stationary solutions considered are all the constants. But in our case the constants are not solutions except if $\operatorname{div}(f(k, x)) = 0$ for all $x \in \mathbb{R}^d$. The existence of another class of stationary solutions is proved by Dalibard (see theorem 2 and lemma 6 in [2]): there exist periodic stationary solutions, indexed by their space average.

In this section, we recall a part of her results for the following equation:

$$\operatorname{div}(f(u,x)) = \Delta u, x \in \mathbb{R}^d$$

where $x \mapsto f(\cdot, x)$ is an Y-periodic function with $Y = \prod_{i=1}^{d}(0, T_i)$ the basis of a lattice. We note the space average of a function u: $\langle u \rangle_Y = \frac{1}{|Y|}\int_Y u(x)dx$.

Theorem 3. *Let $f = f(u,x) \in \mathcal{C}^2(\mathbb{R}, \mathcal{C}^1(\mathbb{R}^d))$ such that $\partial_u f \in L^\infty(\mathbb{R} \times Y)$. Assume that there exist $C_0 > 0$, and $n \in [0, \frac{d+2}{d-2})$ when $d \geq 3$, such that for all $(p,x) \in \mathbb{R} \times Y$*

$$|\operatorname{div} f(p,x)| \leq C_0(1 + |p|^n).$$

Then for all $p \in \mathbb{R}$, there exists a unique solution $w(\cdot, p) \in H^1_{per}(Y)$ of

$$-\Delta w(x,p) + \operatorname{div} f(w(x,p), x) = 0, \text{ such that } \langle w(\cdot, p)\rangle_Y = p.$$

For all $p \in \mathbb{R}, w(\cdot, p)$ belongs to $W^{2,q}_{per}(Y)$ for all $1 < q < \infty$ and for all $R > 0$, there exists $C_R > 0$ such that

$$\|w(\cdot, p)\|_{W^{2,q}(Y)} \leq C_R \quad \forall p \in \mathbb{R}, |p| \leq R,$$

$C_R > 0$ depending only on d, Y, C_0, n, q, p_0 and R.

Furthermore, for all $p \in \mathbb{R}, \partial_p w(\cdot, p) \in H^1_{per}(Y)$ is in the kernel of the linear operator

$$-\Delta + \operatorname{div}(\partial_u f(w(x,p), x)\cdot) = 0, \text{ and } \langle \partial_p w \rangle_Y = 1.$$

And there exists $\alpha > 0$ depending only on d, Y and $\|\partial_u f\|_\infty$ such that

$$\partial_p w(x,p) > \alpha \text{ for a.e. } (x,p) \in Y \times \mathbb{R}.$$

Hence,

$$\lim_{p \to +\infty} \inf_Y w(x,p) = +\infty,$$

$$\lim_{p \to -\infty} \sup_Y w(x,p) = -\infty.$$

Remarks 1.

- A consequence of this theorem is that for all $x \in \mathbb{R}^d$, the application $p \mapsto w(p,x)$ is increasing and bijective from $\mathbb{R}$ to $\mathbb{R}$.

- In this theorem, we impose the restrictive assumption that $\partial_u f \in L^\infty$ on the whole domain $\mathbb{R} \times Y$. When $\partial_u f$ belongs only to $L^\infty_{loc}(L^\infty(Y))$, we obtain that $\partial_p w > 0$ but we have not the existence of the constant α. Hence, we have no result on the limit when $p \to \pm\infty$ of $\inf_Y w(x,p)$ and $\sup_Y w(x,p)$, but we have that the application

$$\begin{array}{ccl} \mathbb{R} & \to & \left]\lim\limits_{p\to+\infty} \inf\limits_Y w(x,p), \lim\limits_{p\to-\infty} \sup\limits_Y w(x,p)\right[\\ p & \mapsto & w(p,x) \end{array}$$

is bijective. And we can adapt the result of theorem 2 in this case : we just have to make the assumption that there exists p such that for all $x \in \mathbb{R}^d, u_0(x) \in [w(-p,x), w(p,x)]$.

In the sequel, we use the notation: $w_p = w(\cdot, p)$.

3 The nonlinear semigroup

In what follows, we focus on the Cauchy problem for equation (1):

$$\left\{ \begin{array}{ll} \partial_t u + \operatorname{div}(f(u,x)) = \Delta u, & \forall t > 0, \forall x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d, \end{array} \right. \tag{3}$$

where the initial datum u_0 belongs to $L^\infty(\mathbb{R}^d)$. First, we adapt the approach of Serre [9] to prove the existence of solutions and their properties: comparison principle, L^1-contraction, conservation of mass. Then, we prove a dispersion inequality, using a new type of entropy based on periodic solutions.

3.1 Existence of the nonlinear semigroup

As in [9], the proof of the existence of solutions is based on Duhamel's formula for heat equation. We also need a comparison principle to replace the maximum principle which is not true here.

Let us write problem (3) in the form:

$$\left\{ \begin{array}{ll} \partial_t u - \Delta u = -\operatorname{div}(f(u,x)), & t > 0, x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d. \end{array} \right. \tag{4}$$

Here, the heat operator appears in the left handside of (4), and the right handside is a lower order perturbation. Denote H^t the heat semigroup and K^t its kernel. They are given by:

$$H^t u_0 = K^t * u_0, \quad K^t(x) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{\|x\|^2}{4t}\right)$$

and satisfy the following properties:

$$\|H^t u_0\|_p \leq \|u_0\|_p, \quad 1 \leq p \leq \infty, \tag{5}$$

$$\|\nabla_x H^t u_0\|_p \leq c'_p t^{-\frac{1}{2}} \|u_0\|_p, \quad 1 \leq p \leq \infty, \tag{6}$$

$$\int_{\mathbb{R}^d} K^t(x)dx = 1, \quad \int_{\mathbb{R}^d} \nabla_x K^t(x)dx = 0. \tag{7}$$

We prove the following result:

Proposition 1. *Assume that $f \in \mathcal{C}^k(\mathbb{R}, \mathcal{C}^1(\mathbb{R}^d))$. Then for all $a \in L^\infty(\mathbb{R}^d)$, there exist $T > 0$ and a unique solution $u \in L^\infty([0,T] \times \mathbb{R}^d)$ of (3). Moreover, $u \in \mathcal{C}^k((0,T), \mathcal{C}^\infty(\mathbb{R}^d))$ and T depends only on $\|u_0\|_\infty$.*

Proof. We are searching for the mild solution of (3), i.e which verifies the Duhamel's formula:

$$\begin{aligned} u(t,\cdot) &= K^t * u_0 - \int_0^t K^{t-s} * \operatorname{div}(f(u(s,\cdot),\cdot))ds \\ &= K^t * u_0 - \int_0^t \nabla_x K^{t-s} * f(u(s,\cdot),\cdot)ds. \end{aligned}$$

Hence, we search for the solution of (3) as a fixed point of the map

$$M : u \mapsto \Big(t \mapsto K^t * u_0 - \int_0^t \nabla_x K^{t-s} * f(u(s,\cdot),\cdot)ds\Big).$$

In order to use Picard's fixed point theorem we need to find a space which is stable by M and where M is contractant. Using (5)-(6) with $p = \infty$ we have the following estimate for all $u \in L^\infty(\mathbb{R}^d)$:

$$\|Mu(t)\|_\infty \leq \|u_0\|_\infty + \int_0^t \frac{c'_\infty}{(t-s)^{\frac{1}{2}}} \|f(u(s,\cdot),\cdot)\|_\infty ds.$$

We assume that for all $0 \leq \ s \leq \ T, \|u(s)\|_\infty \leq 2\|u_0\|_\infty$. Since $f(\cdot,x)$ is locally in L^∞, uniformly in x, there exists C such that for all $0 \leq s \leq T$,

$$\|f(u(s,\cdot),\cdot)\|_\infty \leq C$$

where C does not depend on u, but only on $\|u\|_{L^\infty((0,t)\times\mathbb{R}^d)} \leq 2\|u_0\|_\infty$. Therefore, we obtain the following estimate

$$\|Mu(t)\|_\infty \leq \|u_0\|_\infty + 2c'_\infty C\sqrt{T}, \quad \forall 0 \leq t \leq T.$$

For T sufficiently small ($2c'_\infty C\sqrt{T} < \|u_0\|_\infty$), the map M preserves the ball of radius $2\|u_0\|_\infty$ of $L^\infty((0,T)\times\mathbb{R}^d)$. This ball is denoted $B(2\|u_0\|_\infty)$. Next we prove that M is a contraction: let $u, v \in B(2\|u_0\|_\infty)$, then

$$Mv(t) - Mu(t) = \int_0^t \nabla_x K^{t-s} * (f(u(s,\cdot),\cdot) - f(v(s,\cdot),\cdot))ds.$$

Since $f(\cdot, x)$ is locally Lipschitz, uniformly in x, there exists C' (depending on $2\|u_0\|_\infty$) such that $\|f(u,\cdot) - f(v,\cdot)\|_\infty \leq C'\|u-v\|_\infty$. Hence, we obtain

$$\|Mu - Mv\|_\infty \leq 2c'_\infty C'\sqrt{T}\|u-v\|_\infty$$

and for T small enough, the map M is stable and contractant on $B(2\|u_0\|_\infty)$. We can now use Picard's fixed point theorem to obtain a unique local solution in $L^\infty([0,T]\times\mathbb{R}^d)$. Moreover, using again Duhamel's formula, we prove that this solution is regular in time if f is regular in u and x ; for instance u is in $\mathcal{C}^k((0,T),\mathcal{C}^\infty(\mathbb{R}^d))$ if f is in $C^k(\mathbb{R},\mathcal{C}^1(\mathbb{R}^d))$. ◻

To prove global existence in homogeneous problem, one uses maximum principle. When the problem is inhomogeneous, this maximum principle is false and one uses a comparison principle:

Lemma 1. Comparison principle: *Let $u, v \in L^\infty([0,T]\times\mathbb{R}^d)$ two solutions of (1) on $(0,T)$ such that for all $x \in \mathbb{R}^d, u_0(x) \leq v_0(x)$. Then for all $t \in [0,T]$, and $x \in \mathbb{R}^d$, we have $u(t,x) \leq v(t,x)$.*

Using this lemma, we then prove global existence of solution:

Proposition 2. *Assume that $f \in \mathcal{C}^k(\mathbb{R},\mathcal{C}^1(\mathbb{R}^d))$. Then for all $u_0 \in L^\infty(\mathbb{R}^d)$, there exists a unique solution $u \in \mathcal{C}^k(\mathbb{R},\mathcal{C}^\infty(\mathbb{R}^d))$ of (3).*

Proof. From theorem 3 and the remark 1 we deduce that for all x, the application $p \mapsto w_p(x)$ is invertible from $\mathbb{R}$ to $\mathbb{R}$. Since $u_0 \in L^\infty(\mathbb{R}^d)$, there exists p such that $w_{-p}(x) \leq u_0(x) \leq w_p(x)$. Proposition 1 gives us T (we can chose $T = T(\max\{\|w_{-p}\|_\infty, \|w_p\|_\infty\})$) and a unique solution u. The lemma implies that for all $t \in (0,T)$, and $x \in \mathbb{R}$, we have $w_{-p}(x) \leq u(t,x) \leq w_p(x)$. Therefore, we can iterate the local existence to prove that u exists on $(0,T), \ldots, (kT,(k+1)T)$ for any $k \in \mathbb{N}$. Finally, we obtain a unique bounded solution, global and smooth for positive time. ◻

Next, we define the nonlinear semigroup S^t on $L^\infty(\mathbb{R}^d)$. From now, we will note $u = S^t u_0, v = S^t v_0$ if $u_0, v_0 \in L^\infty(\mathbb{R}^d)$.

As in [9], we have some properties on this semigroup: we have already mentioned the comparison principle (lemma 1). We also have L^1-contraction and conservation of mass. And as said above, the constants are no longer stationary solutions: they are replaced by periodic functions.

Proposition 3. *For all $u_0, v_0 \in L^\infty(\mathbb{R}^d)$ such that $u_0 - v_0 \in L^1(\mathbb{R}^d)$, for all $t > 0$ we have*

i) L^1-contraction : $S^t u_0 - S^t v_0 \in L^1(\mathbb{R}^d)$ *and* $\|S^t u_0 - S^t v_0\|_1 \leq \|u_0 - v_0\|_1$;

ii) conservation of mass : $\displaystyle\int_{\mathbb{R}^d} (S^t u_0 - S^t v_0) = \int_{\mathbb{R}^d} (u_0 - v_0)$.

Proof. Let $u_0, v_0 \in L^\infty(\mathbb{R}^d)$ such that $u_0 - v_0 \in L^1(\mathbb{R}^d)$. We first prove that $S^t u_0 - S^t v_0 \in L^1(\mathbb{R}^d)$. Using Duhamel's formula, one obtains:

$$v(t) - u(t) = K^t * (v_0 - u_0) - \int_0^t (\nabla_x K^{t-s}) * (f(v(s,\cdot),\cdot) - f(u(s,\cdot),\cdot))ds. \quad (8)$$

Taking the L^1-norm and using estimates (5)-(6) for $p = 1$, we deduce that

$$\sup_{s \leq t} \|v(s) - u(s)\|_1 \leq \|v_0 - u_0\|_1 + 2c_1' C' \sqrt{t} \sup_{s \leq t} \|v(s) - u(s)\|_1.$$

Hence, for t small enough, $v(s) - u(s) \in L^1(\mathbb{R}^d)$, for all $0 \leq s \leq t$ and by induction it is true for all $t \in \mathbb{R}^+$.
We now prove the L^1-contraction principle. For all $u_0, v_0 \in L^\infty(\mathbb{R}^d)$ one shows that

$$\partial_t |u - v| + \operatorname{div}(\operatorname{sgn}(u - v)(f(u,\cdot) - f(v,\cdot))) \leq \Delta |u - v|.$$

Noting

$$w = -K^t * |v_0 - u_0| + \int_0^t \partial_x K^{t-s} * \operatorname{div}((f(u,x) - f(v,x)) \operatorname{sgn}(u - v)) + |u - v|, \quad (9)$$

we easily prove $\partial_t w \leq \Delta w$ and $w(0) = 0$. Using comparison principle, we have $w \leq 0$. We integrate (9) according to x to obtain

$$0 \geq \int_{\mathbb{R}^d} w = -\int_{\mathbb{R}^d} |v_0 - u_0| + \int_{\mathbb{R}^d} |u - v|. \quad (10)$$

From (10), we deduce the contraction principle.
Let us now prove the conservation of mass. Integrating (8), and using (7) we immediately obtain for all $u_0, v_0 \in L^\infty(\mathbb{R}^d)$: $\partial_t \int_{\mathbb{R}^d}(u-v) = 0$ and

$$\int_{\mathbb{R}^d}(u-v) = \int_{\mathbb{R}^d}(u_0 - v_0).$$

□

3.2 Dispersion inequality

In this section, we prove the following dispersion inequality for equation (1):

Proposition 4. *Let $R \in \mathbb{R}$. There exists $C > 0$ so that for all $p \in \mathbb{R}, b \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ such that $w_{-R} \leq w_p + b \leq w_R$, $u(t) = S^t(w_p + b)$ verifies a dispersion inequality:*

$$\|u(t) - w_p\|_2 \leq C_d \frac{\|b\|_1}{t^{d/4}}. \tag{11}$$

This estimate gives convergence in L^2 when $u_0 - w_0 \in L^1(\mathbb{R}^d)$ and the speed of this convergence. In section 4, we will see how L^2-convergence imply L^1-convergence in the one dimensional space case.

This property is first proved by Bénilan and Abourjaily in [1] in the case where f does not depend on x. When $\tilde{S}^t$ denotes the semigroup of (2), their result can be written as follows:

$$\|\tilde{S}^t u_0\|_2 \leq c_d \frac{\|u_0\|_1}{t^{d/4}}.$$

In this case, the proof of the inequality is based on the fact that for all convex function η, there exists g such that for all u, $\eta'(u)\,\mathrm{div}(f(u)) = \mathrm{div}(g(u))$, in particular for $\eta(u) = u^2$. This property is false in our case but we still have a dispersion inequality (11).

To prove proposition 4, we use a new class of entropies. When f does not depend on x, an interesting class of entropies is the Kružkov entropies $u \mapsto |u-k|$ with $k \in \mathbb{R}$. Those are convex functions and for all u solution of (2), we have the inequality

$$\partial_t|u-k| + \mathrm{div}(\mathrm{sgn}(u-k)(f(u) - f(k))) \leq \Delta|u-k|.$$

This inequality is still true in our case but we do not want to compare our solutions to constants anymore, because they are not stationary solutions of

(3). Hence, we define a new type of entropy, using the stationary solutions w_p.

Proof. Without loss of generality we assume that $p = 0$. We have just said that we need to base our new entropy on the stationary solutions. Theorem 3 gives us that for all $p \in \mathbb{R}$, there exists a unique stationary solution w_p under the constraint $\langle w_p \rangle_Y = p$. Following the construction of Kružkov entropies, let us consider , for any $p \in \mathbb{R}$, the function η_p such that

$$\eta_p : (x, u) \mapsto \eta_p(x, u) = |u - w_p(x)|.$$

This application verifies the inequality:

$$\partial_t \eta_p(u(t, x), x) + \operatorname{div}(\operatorname{sgn}(u - w_p)(f(u, x) - f(w_p, x))) \leq \Delta \eta_p.$$

In order to define our new entropy η, we define two auxiliary functions $p(u, x)$ and $\pi(x, t)$. We recall that for all $x \in \mathbb{R}^d$, the function $p \mapsto w_p(x)$ is a bijection from $\mathbb{R}$ to $\mathbb{R}$. We note $p(u, x)$ the inverse of this application. It verifies:

$$\forall x \in \mathbb{R}^d, u \in \mathbb{R}, w_{p(u,x)}(x) = u.$$

If u is a function defined on $\mathbb{R}^+ \times \mathbb{R}^d$ such that for all (t, x), we define $\pi(t, x) = p(u(t, x), x)$. One remarks that $-R \leq \pi \leq R$. We can now define our particular entropy η as:

$$\eta(u, x) = \int_0^{p(u,x)} (u - w_p(x)) dp.$$

This function is non negative. Next, we derive energy estimate on u using this new entropy. Deriving $\eta(u(t, x), x)$ with respect to t and using (3), one obtains

$$\partial_t(\eta(u(t, x), x)) = \int_0^{\pi(t,x)} \Delta(u - w_p) dp - \int_0^{\pi(t,x)} \operatorname{div}(f(u, x) - f(w_p, x)) dp. \quad (12)$$

The last term of (12) is written as:

$$\int_0^{\pi(t,x)} \operatorname{div}(f(u, x) - f(w_p, x)) dp = \operatorname{div} \left(\int_0^{\pi(t,x)} (f(u, x) - f(w_p, x)) dp \right)$$

and

$$\int_0^{\pi(t,x)} \Delta(u-w_p)dp = \Delta(\eta(u(t,x),x)) - \nabla\pi\cdot\nabla(u-w_p)|_{p=\pi(t,x)}.$$

We then obtain the following partial differential equation:

$$\partial_t\eta(u)+\mathrm{div}\left(\int_0^{\pi(t,x)}(f(u,x)-f(w_p,x))dp\right)=\Delta\eta(u)-\nabla\pi\nabla(u-w_p)|_{p=\pi(t,x)}. \quad (13)$$

Moreover, we have the equality:

$$0=\nabla(u(t,x)-w_{\pi(t,x)}(x)) = \nabla(u-w_p)|_{p=\pi(t,x)} - \partial_p w_\pi\cdot\nabla\pi. \quad (14)$$

We deduce from (13) and (14) that η satisfies the equation:

$$\partial_t\eta(u)+\mathrm{div}\left(\int_0^{\pi(t,x)}(f(u,x)-f(w_p,x))dp\right) = \Delta\eta - \partial_p w_\pi\cdot|\nabla\pi|^2. \quad (15)$$

Integrate equation (15) in space: we get

$$\frac{d}{dt}\int_{\mathbb{R}^d}\eta(u)(x)dx + \int_{\mathbb{R}^d}\partial_p w_\pi|\nabla\pi|^2 = 0.$$

Moreover, theorem 3 gives us $\partial_p w_\pi \geq \alpha > 0$. Using this inequality and Nash inequality ([10]):

$$\|\pi\|_2 \leq c_d\|\pi\|_1^{(1-\theta)}\|\nabla\pi\|_2^\theta \quad \text{where} \quad \frac{1}{\theta}=1+\frac{2}{d},$$

we obtain:

$$\frac{d}{dt}\int_{\mathbb{R}^d}\eta(u)(x)dx + C_d\frac{\|\pi\|_2^{2/\theta}}{\|\pi\|_1^{2(1-\theta)/\theta}} \leq 0. \quad (16)$$

Let us now relate π with η:

$$\eta(u(t,x),x) = \int_0^{\pi(t,x)}(u(t,x)-w_p(x))dp.$$

From the estimate

$$\begin{aligned} |u(t,x)-w_p(x)| &= |w_{\pi(t,x)}(x)-w_p(x)| = \left|\int_p^{\pi(t,x)}\partial_p w_p(x)dp\right| \\ &\leq |\pi(t,x)|\sup_p|\partial_p w_p|, \end{aligned} \quad (17)$$

we deduce,

$$\eta(u(t,x),x) \leq |\pi(t,x)|^2 \sup_p |\partial_p w_p|.$$

Since $\partial_p w_p$ is locally bounded in p, i.e. $\partial_p w_p(x) \leq C$ for all $x \in \mathbb{R}^d$, for all $p \in [-R,R]$, we deduce the inequality:

$$\eta(u(t,x),x) \leq C|\pi(t,x)|^2. \tag{18}$$

We combine (16) and (18) to obtain:

$$\frac{d}{dt}\left(\int_{\mathbb{R}^d} \eta(u)(x)dx\right) + C\frac{(\int_{\mathbb{R}^d} \eta(u)(x)dx)^{1/\theta}}{\|\pi\|_1^{2(1-\theta)/\theta}} \leq 0.$$

We have now to overvalue $\|\pi\|_1$ uniformly in t. Now

$$\pi(t,x) = p(u(t,x),x) - p(w_0(x),x) = \int_{w_0(x)}^{u(t,x)} \partial_u p(w,x)dw.$$

We deduce from the minoration $\partial_p w_p \geq \alpha$ the estimate $\partial_u p \leq 1/\alpha$ and we deduce:

$$\|\pi(t)\|_1 \leq \frac{1}{\alpha}\|u(t) - w_0\|_1 \leq \frac{1}{\alpha}\|b\|_1$$

with L^1-contraction. Finally, we have the inequation

$$\frac{d}{dt}\left(\int_{\mathbb{R}^d} \eta(u)(x)dx\right) + \frac{C}{\|b\|_1^{2(1-\theta)/\theta}}\left(\int_{\mathbb{R}^d} \eta(u)(x)dx\right)^{1/\theta} \leq 0. \tag{19}$$

Using $g := -\left(\int_{\mathbb{R}^d} \eta(u)(x)dx\right)^{1-1/\theta}$, we solve this inequation and we obtain

$$g(t) \leq (1-1/\theta)C\frac{t}{\|b\|_1^{2(1-\theta)/\theta}}.$$

Hence,

$$\left(\int_{\mathbb{R}^d} \eta(x)dx\right) \leq C'\frac{\|b\|_1^2}{t^{d/2}}.$$

To conclude the proof, we prove that there exists $C > 0$ such that for all $t \geq 0$, $\sqrt{\int \eta(u(t))(x)} \geq C\|u(t) - w_0\|_2$. First, we have

$$\begin{aligned}
\eta(u)(x) &= \int_0^{p(u(x),x)} (u(x) - w_p(x))dp \\
&= \int_0^{p(u(x),x)} (\int_p^{p(u(x),x)} \partial_p w_q(x)dq)dp \\
&\geq \alpha \int_0^{p(u(x),x)} (p(u(x),x) - p)dp \\
&= \alpha\frac{p(u(x),x)^2}{2}.
\end{aligned}$$

Then, estimate (17) shows that:

$$|u - w_0|^2 \leq (\sup_p |\partial_p w_p|)^2 p(u(x), x)^2 \leq C^2 p(u(x), x)^2.$$

This concludes the proof of the theorem. $\square$

4 One-dimensional space case: proof of theorem 2

In this section, we prove L^1-convergence in one space dimension. We bypass the utilisation of Duhamel's formula by counting the number of sign changes. This method is used by Freistühler and Serre in [3] to prove that constants are stable in L^1 when the flux f does not depend on x, and when the space dimension is one. It uses a lemma of Matano [4] which gives an evaluation of the number of sign changes for the solution of our equation. The proof is carried out in four steps: (1) At first, we make additional assumptions on f and the initial datum. (2) Then, we prove L^2-estimates on u and its primitive V and we deduce that $\|V(t)\|_\infty$ vanishes at $+\infty$. (3) Using lemma of Matano, we find that $\|u(t)\|_1$ is controlled by $\|V(t)\|_\infty$, so we prove the result under the additional hypothesis. (4) We generalized the result without these assumptions.

Proof. First, up to a translation, we will assume that

$$p = 0, w_p \equiv 0 \text{ and } f(0, \cdot) \equiv 0.$$

We define $F(u, x) = f(u, x) - \partial_u f(0, x)u$ which verifies: $F(0, \cdot) \equiv 0$, and $\partial_u F(0, \cdot) \equiv 0$ and we deduce the inequality

$$F(u, x) \leq \frac{|u|^2}{2} \sup |\partial_u^2 F|.$$

(1) Let us first assume that b is bounded in the following sense: let

$$p^+ = \min\{p : b \leq w_p\}, p^- = \max\{p : b \geq w_p\},$$

we assume that

$$\max\{\|w_{p^+}\|_\infty, \|w_{p^-}\|_\infty\} < r.$$

Therefore, we have: $|b| < r$ and using the comparison property for all t, $|S^t b| < r$. Moreover, we assume $\|b\|_1 \sup_{[-r,r]} |\partial_u^2 F| \leq 1$. We will see at the end of the proof how to remove these assumptions.
We further assume that $b \in \mathcal{C}_0^\infty(\mathbb{R}, [-r,r]), l(b) < \infty$ where $l(b)$ is the number of sign changes of b. Actually, we can approximate every function b that verifies the conditions of step 1 by a function in $\mathcal{C}_0^\infty$; and since the support is compact, we can suppose that the sign of the function changes only a finite number of time.
(2) Assume now that b verifies all the previous assumptions. We define $V(x) = \int_{-\infty}^x u(t,y)dy$. Since u belongs to L^1, V is well defined and belongs to L^∞ and $\|V\|_\infty \leq \|b\|_1$. Moreover, since $\int_\mathbb{R} b = 0$ and we have mass conservation, we have that $V \in \mathcal{C}_0^\infty$. In search of estimates on V, we consider an equation verified by V:

$$\partial_t V + \partial_u f(0,x)\partial_x V + F(\partial_x V, x) = \partial_x^2 V. \tag{20}$$

Let $\theta : x \mapsto \theta(x)$ from $\mathbb{R}$ to $\mathbb{R}$ be a positive function (which will be specified later). Multiplying by θV and integrating in space, we obtain:

$$\frac{d}{dt}\int \frac{1}{2}\theta V^2 + \int \theta |\partial_x V|^2 = -\int \theta V F(\partial_x V, x) + \int \frac{V^2}{2}\left(\partial_x(\theta \partial_u f(0,x))\partial_x^2\theta\right).$$

Besides, we have the inequality: $|F(\partial_x V, x)| \leq \frac{|\partial_x V|^2}{2} \sup |\partial_u^2 F|$. We deduce the estimate:

$$\frac{d}{dt}\left(\int \theta V^2\right) \leq -\int \theta |\partial_x V|^2 + \int V^2 \left(\partial_x(\theta \partial_u f(0,x)) + \partial_x^2 \theta\right).$$

Now we choose θ to obtain an estimate on $\int \theta V^2$. We impose:

- $\theta > \alpha > 0$ so that $V \mapsto \int \theta V^2$ is a norm on L^2.
- $\partial_x(\theta \partial_u f(0,x)) + \partial_x^2 \theta = 0$.
  Actually, we only need that $\partial_x(\theta \partial_u f(0,x)) + \partial_x^2\theta \leq 0$.

The following lemma ensures the existence of such a θ:

Lemma 2. *There exists $\theta > 0$ in $H^1_{per}(Y)$ such that*

$$\partial_x(\theta \partial_u f(0,x)) + \partial_x^2 \theta = 0.$$

Proof. We focus on the equation:

$$\partial_t w - \partial_x(f(w,x)) = \partial_x^2 w.$$

Theorem 3 ensures the existence of a periodic stationary solution $\tilde{w}_p$ of space average p and this one verifies: $\partial_p \tilde{w}_p > 0$. Moreover, the function defined by $\theta \equiv \partial_p \tilde{w}_p|_{p=0}$ is Y-periodic, in H^1 and verifies the following equation:

$$\partial_x(\theta \partial_v f(\tilde{w}_0, x)) + \partial_x^2 \theta = 0.$$

We remark that $\partial_x f(0,x) = 0 = \partial_x^2 0$. Since $\tilde{w}_0$ is the unique function such that $\partial_x^2 \tilde{w}_0 = -\partial_x f(\tilde{w}_0, x)$ and $\langle \tilde{w}_0 \rangle_Y = 0$, we have $\tilde{w}_0 \equiv 0$. $\square$

The definition of θ ensures the inequality:

$$\frac{d}{dt}\left(\int \theta V^2\right) \leq -\int \theta |\partial_x V|^2.$$

Since θ belongs to $H^1_{per}(Y) \subset \mathcal{C}(\mathbb{R})$, there exists $c > 0$ such that $c < \theta$. Hence, we deduce that V is bounded in $L^2(\mathbb{R})$:

$$c \int |V|^2(t) \leq \int \theta |V|^2(t) \leq \int \theta |V|^2(0). \tag{21}$$

We also have an estimate on $\|u\|_2$. Indeed, we proved in section 4 the dispersion inequality (11) for u:

$$\left(\int_{\mathbb{R}} |u(x,t)|^2 dx\right) \leq C_1 \frac{\|b\|_1^2}{t^{1/2}}.$$

We deduce that

$$\lim_{t\to\infty} \|u(t)\|_2 = 0. \tag{22}$$

We can now prove an estimate on $\|V\|_\infty$. We have

$$V^2(x,t) = 2\int_{-\infty}^{x} u(y,t)V(y,t)dy \leq 2\|u(\cdot,t)\|_2 \|V(\cdot,t)\|_2.$$

From equations (22) and (21), we deduce:

$$\lim_{t\to\infty} \|u(\cdot,t)\|_2 = 0, \|V(\cdot,t)\|_2 \text{ uniformly bounded in } t.$$

Consequently $\lim\limits_{t\to\infty} \|V(\cdot,t)\|_\infty = 0$.
(3) We now need an estimate on the number of sign changes of the solution u. To obtain it, we refer to the article of Matano [4] in which an estimate on the lap number of a solution of a parabolic problem is proved.
Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. We define its *lap number* l as the supremum of 0 and all $k \in \mathbb{N}$ with the property that there exist $k+1$ points $x_0 < \cdots < x_k$ such that

$$\forall 0 < i < k, (g(x_{i+1}) - g(x_i))(g(x_i) - g(x_{i-1})) < 0.$$

We adapt the lemma of Matano [4] to get:

Lemma 3. *For any bounded solution $V : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ of (20):*

$$\partial_t V + \partial_u f(0,x)\partial_x V + F(\partial_x V, x) = \partial_x^2 V$$

with $V(0,\cdot) \in \mathcal{C}_0^\infty(\mathbb{R})$ having a finite lap number, the lap number of $V(t,\cdot)$ is uniformly bounded for all $t \geq 0$.

To do that, we just have to notice that $F(\partial_x V, x) = \tilde{F}(\partial_x V, x)\partial_x V$ with $\tilde{F}(\partial_x V, x)$.
Since the number of sign changes of b is finite, $V(0,x)$ has a finite lap number. The lemma of Matano proves that $\forall t, \exists \xi_1^t, \dots, \xi_m^t$ such that V is monotone on $]-\infty = \xi_0^t; \xi_1^t[, \dots,]\xi_m^t; \xi_{m+1}^t = \infty[$. Therefore, the sign of u does not change on the same intervals. We deduce:

$$\begin{array}{rcl} \|u(\cdot,t)\|_1 = \sum_{i=0}^m \left| \int_{\xi_i^t}^{\xi_{i+1}^t} u(x,t)dx \right| & = & \sum_{i=0}^m |V(\xi_{i+1}^t, t) - V(\xi_i^t, t)| \\ & \leq & 2(m+1)\|V(t)\|_\infty \to 0. \end{array}$$

Therefore the theorem is proved under the assumptions:

$$\max\{\|w_{p^+}\|_\infty, \|w_{p^-}\|_\infty\} < r, \quad \|b\|_1 \sup_{[-r,r]} |\partial_u^2 F| \leq 1$$

with

$$p^+ = \min\{p : b \leq w_p\}, \quad p^- = \max\{p : b \geq w_p\}.$$

(4) Next, we show how to remove these assumptions. We define

$$A_p = \left\{ b \in L^1(\mathbb{R}) : \int_{-\infty}^{\infty} b = 0 \text{ et } \forall x, w_{-p}(x) \leq b(x) \leq w_p(x) \right\}.$$

We note $M_p = \max\{\|w_{-p}\|_\infty, \|w_p\|_\infty\}$. Hence, we have

$$\sup_{[-M_p,M_p]} |\partial_u^2 F| < \infty.$$

Let now $b \in A_p$ et $n = 2\|b\|_1 \sup_{[-M_p,M_p]} |\partial_v^2 F|$. Using $w_{-p} \le 0 \le w_p$, we have $b/n \in A_p$ et $\forall k \in \{1,\ldots,n\}, \frac{kb}{n} \in A_p$. The properties of the nonlinear semigroup show that A_p is stable under S^t, so we have for all $t, S^t(\frac{kb}{n}) \in A_p$. By induction on k, we can prove the theorem for $\frac{kb}{n}$. Let P_k the property:

$$P_k : \lim_{t\to\infty} \left\| S^t\left(\frac{kb}{n}\right)\right\|_1 = 0$$

P_1: We have $b/n \in A_p, \|\frac{b}{n}\|_1 \sup_{[-M_p,M_p]} |\partial_u^2 F| = \frac{1}{2} < 1$. Using step 3, we deduce: $\lim_{t\to\infty} \left\| S^t\left(\frac{b}{n}\right)\right\|_1 = 0$.
P_k: Let assume that P_k with $k < n$ is true and let prove P_{k+1}. We have $S^t((k+1)\frac{b}{n}) \in A_p$. Moreover, the L^1-contraction property gives:

$$\left\| S^t\left((k+1)\frac{b}{n}\right) - S^t\left(\frac{kb}{n}\right)\right\|_1 \le \left\|\frac{b}{n}\right\|_1.$$

We deduce:

$$\|S^t((k+1)\frac{b}{n})\|_1 \le \|S^t(\frac{kb}{n})\|_1 + \|\frac{b}{n}\|_1.$$

Since

$$\lim_{t\to\infty} \left\| S^t\left(\frac{kb}{n}\right)\right\|_1 = 0,$$

we have

$$\|S^t((k+1)\frac{b}{n})\|_1 \sup_{[-M_p,M_p]} |\partial_u^2 F| < 1$$

for t large enough. Furthermore, $S^t((k+1)\frac{b}{n}) \in A_p$. Hence, we can use the conclusion of step 3 again to conclude the proof. $\square$

5 Perspectives

In this paper we have proved the L^1-stability of the periodic stationary solutions of (1) in the one-dimensional space case. The proof uses a dispersion inequality which is also verified in the multidimension space case and the

lemma of Matano (lemma 3) about the number of sign changes of the solution of (1). But in the multidimension space case, the lemma of Matano has no more sense. An idea to bypass it is to use Duhamel's formula, as done by Serre in [9]. In this purpose, we consider the linearized operator $L = \Delta - \mathrm{div}(\partial_u f(0,x)\cdot)$, and we write the equation in the form:

$$(\partial_t - L)u = -\,\mathrm{div}(F(u,x))$$

with $F(u,x) = f(u,x) - \partial_u f(0,x)u$. We note $\tilde{K}^t$ the kernel of the operator L so that we obtain Duhamel's formula:

$$u(t) = \tilde{K}^t * b - \int_0^t \nabla_x \tilde{K}^{t-s} * F(u(s,\cdot),\cdot)ds.$$

Taking L^1-norms:

$$u(t) \leq \|\tilde{K}^t * b\|_1 + \int_0^t \|\nabla_x \tilde{K}^{t-s}\|_1 \|F(u(s,\cdot),\cdot)\|_1 ds. \tag{23}$$

Moreover, we have $\partial_u F(0,\cdot) \equiv 0$, so we obtain $|F(u,\cdot)| \leq |u|^2$. Hence, dispersion inequality (11) gives:

$$\|F(u(s,\cdot),\cdot)\|_1 \leq C_d^2 \frac{\|b\|_1^2}{s^{d/2}}.$$

To obtain an L^1-convergence theorem similar to theorem 2, we can use estimates on the kernel $\tilde{K}^t$ and its derivative $\nabla_x \tilde{K}^t$. Some results on this kernel are given by Oh and Zumbrun in [5] and [6] when the space dimension is one. When the space dimension d is larger than 2, we can refer to [7] and [8] in which they obtain large-time estimates in L^q where $q \geq 2$, and when f is periodic in only one direction. But, until now, we have not large-time L^1-estimates for $d \geq 2$.

To conclude, we can see how estimates can give a theorem: if we obtain suitable estimates, we can bound all the term in (23) by $\|b\|_1^2$ as in [9] and conclude as Serre does by continuity of the limit: $l_0(b) = \lim\limits_{t\to\infty} \|S^t b\|_1$.

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